Noncyclic subgroups (D+F 2.4)

Cyclic subgroups of a group are those generated by a single element of a group, but we can also look at subgroups generated by multiple elements.

Let $A \subseteq G$ be a subset of G. The subgroup generated by A is defined as the smallest subgroup of G containing A. More precisely:

Def: Define $\langle A \rangle = \bigcap_{\substack{A \subseteq H \\ H \leq G}} H$. This is called the subgroup of G generated by A.

Recall from the HW that the intersection of a collection of subgroups is again a subgroup, so $\langle A \rangle \leq G$.

Note that this is very different from how we defined $\langle a \rangle$ for a single element $a \in G$. In fact, these are the same.

Denote $\overline{A} = \{a_1^{k_1} \cdots a_n^{k_n} \mid a_i \in A,\ k_i \in \mathbb{Z},\ n \geq 0\}$, where the A_i are not necessarily distinct.

Claim: $\langle A \rangle = \overline{A}$.

Pf: If $H \supseteq A$ is a subgroup of G, then $a_1^{k_1} \cdots a_n^{k_n} \in H$, since H is closed under the binary operation. Thus,

$\bar{A} \subseteq \langle A \rangle$.

$A \subseteq \bar{A}$, so we just need to show $\bar{A} \leq G$. $1 \in \bar{A}$, so it's nonempty.

$a_1^{k_1} \cdots a_n^{k_n}$, $b_1^{l_1} \cdots b_m^{l_m} \in \bar{A}$. Then

$$(a_1^{k_1} \cdots a_n^{k_n})(b_1^{l_1} \cdots b_m^{l_m})^{-1} = a_1^{k_1} \cdots a_n^{k_n} b_m^{-l_m} \cdots b_1^{-l_m} \in \bar{A},$$

so $\bar{A}$ is a subgroup that contains A, so

$$\langle A \rangle \subseteq \bar{A} \Rightarrow \langle A \rangle = A. \quad \square$$

Ex: In S_3, what's $\langle (1\,2), (2\,3) \rangle$?

$$(1\,2)^2 = 1 = (2\,3)^2$$

$$(1\,2)(2\,3) = (1\,2\,3)$$

But then $\langle (1\,2), (2\,3) \rangle$ has order at least 4, and it divides 6,

so $\langle (12), (23) \rangle = S_3$.

Ex: What is $\langle s, r^2 \rangle$ in D_8?

$\{s, r^2, sr^2, 1\} \leq D_8$ and it contains $\{s, r^2\}$.

It's the smallest such subgroup since $sr^2 = s^1 (r^2)^1$.

$\langle s, r^2 \rangle = \{s, r^2, sr^2, 1\}$.

For abelian groups it's usually easier to compute subgroups generated by sets of elements, since every elt of $\langle A \rangle$ can be expressed as $a_1^{k_1} \cdots a_n^{k_n}$, the a_i distinct (by commuting a_i's).

Lattice subgroups

Now that we know how to describe subgroups of groups in terms of their generators, we show how to depict the relationships among subgroups using a graph.

We draw the lattice of subgroups of a group G as follows:

Each subgroup of G corresponds to a vertex, 1 is on the bottom, G is on the top.

We connect subgroups H and H' by an edge if $H \leq H'$, but there is no K s.t. $K \neq H$ or H' and $H \leq K \leq H'$.

Ex:

1.) $\mathbb{Z}/8\mathbb{Z}$ has subgroups $0, \langle 2 \rangle, \langle 4 \rangle, \mathbb{Z}/8\mathbb{Z}$.

$$\begin{tikzcd}
\mathbb{Z}/8\mathbb{Z} \arrow[d, dash] \\
\langle 2 \rangle \arrow[d, dash] \\
\langle 4 \rangle \arrow[d, dash] \\
0
\end{tikzcd}$$

2.) What are the subgroups of D_8? They can only have order 1, 2, 4, or 8.

The elts of order 2 are r^2, s, sr, sr^2, sr^3.

The elts of order 4 are r, r^3.

The subgroup lattice looks like:

$$
\begin{tikzcd}
 & & D_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & \langle s, r^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \langle r \rangle \arrow[d, dash] & \langle rs, r^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle s \rangle \arrow[drr, dash] & \langle r^2 s \rangle \arrow[dr, dash] & \langle r^2 \rangle \arrow[d, dash] & \langle rs \rangle \arrow[dl, dash] & \langle r^3 s \rangle \arrow[dll, dash] \\
 & & 1 & &
\end{tikzcd}
$$

3.) Q_8 is more straightforward, since i, j, k all have order 4

$$
\begin{tikzcd}
 & Q_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle i \rangle \arrow[dr, dash] & \langle j \rangle \arrow[d, dash] & \langle k \rangle \arrow[dl, dash] \\
 & \langle -1 \rangle \arrow[d, dash] & \\
 & 1 &
\end{tikzcd}
$$