

## Noncyclic subgroups (D+F 2.4)

Cyclic subgroups of a group are those generated by a single element of a group, but we can also look at subgroups generated by multiple elements.

Let  $A \subseteq G$  be a subset of  $G$ . The subgroup generated by  $A$  is defined as the smallest subgroup of  $G$  containing  $A$ . More precisely:

Def: Define  $\langle A \rangle = \bigcap_{\substack{A \subseteq H \\ H \leq G}} H$ . This is called the subgroup of  $G$

generated by  $A$ .

Recall from the HW that the intersection of a collection of subgroups is again a subgroup, so  $\langle A \rangle \leq G$ .

Note that this is very different from how we defined  $\langle a \rangle$  for a single element  $a \in G$ . In fact, these are the same.

Denote  $\bar{A} = \{a_1^{k_1} \dots a_n^{k_n} \mid a_i \in A, k_i \in \mathbb{Z}, n \geq 0\}$ , where the  $A_i$  are not necessarily distinct.

Claim:  $\langle A \rangle = \bar{A}$ .

Pf: If  $H \supseteq A$  is a subgroup of  $G$ , then  $a_1^{k_1} \dots a_n^{k_n} \in H$ , since  $H$  is closed under the binary operation. Thus,

$$\bar{A} \subseteq \langle A \rangle.$$

$A \subseteq \bar{A}$ , so we just need to show  $\bar{A} \leq G$ .  $1 \in \bar{A}$ , so it's nonempty.

$a_1^{k_1} \dots a_n^{k_n}, b_1^{l_1} \dots b_m^{l_m} \in \bar{A}$ . Then

$$(a_1^{k_1} \dots a_n^{k_n}) (b_1^{l_1} \dots b_m^{l_m})^{-1} = a_1^{k_1} \dots a_n^{k_n} b_1^{-l_1} \dots b_m^{-l_m} \in \bar{A},$$

so  $\bar{A}$  is a subgroup that contains  $A$ , so

$$\langle A \rangle \subseteq \bar{A} \Rightarrow \langle A \rangle = \bar{A}. \quad \square$$

Ex: In  $S_3$ , what's  $\langle (12), (23) \rangle$ ?

$$(12)^2 = 1 = (23)^2$$

$$(12)(23) = (123)$$

But then  $\langle (12), (23) \rangle$  has order at least 4, and it divides 6,

so  $\langle (12), (23) \rangle = S_3$ .

Ex: What is  $\langle s, r^2 \rangle$  in  $D_8$ ?

$\{s, r^2, sr^2, 1\} \leq D_8$  and it contains  $\{s, r^2\}$ .

It's the smallest such subgroup since  $sr^2 = s'(r^2)'$ .

$$\langle s, r^2 \rangle = \{s, r^2, sr^2, 1\}.$$

For abelian groups it's usually easier to compute subgroups generated by sets of elements, since every elt of  $\langle A \rangle$  can be expressed as  $a_1^{k_1} \dots a_n^{k_n}$ , the  $a_i$  distinct (by commuting  $a_i$ 's).

## Lattice subgroups

Now that we know how to describe subgroups of groups in terms of their generators, we show how to depict the relationships among subgroups using a graph.

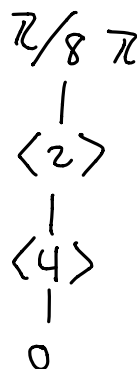
We draw the lattice of subgroups of a group  $G$  as follows:

Each subgroup of  $G$  corresponds to a vertex,  $1$  is on the bottom,  $G$  is on the top.

We connect subgroups  $H$  and  $H'$  by an edge if  $H \leq H'$ , but there is no  $K$  s.t.  $K \neq H$  or  $H'$  and  $H \leq K \leq H'$ .

Ex:

1.)  $\mathbb{Z}/8\mathbb{Z}$  has subgroups  
 $0, \langle 2 \rangle, \langle 4 \rangle, \mathbb{Z}/8\mathbb{Z}$ .

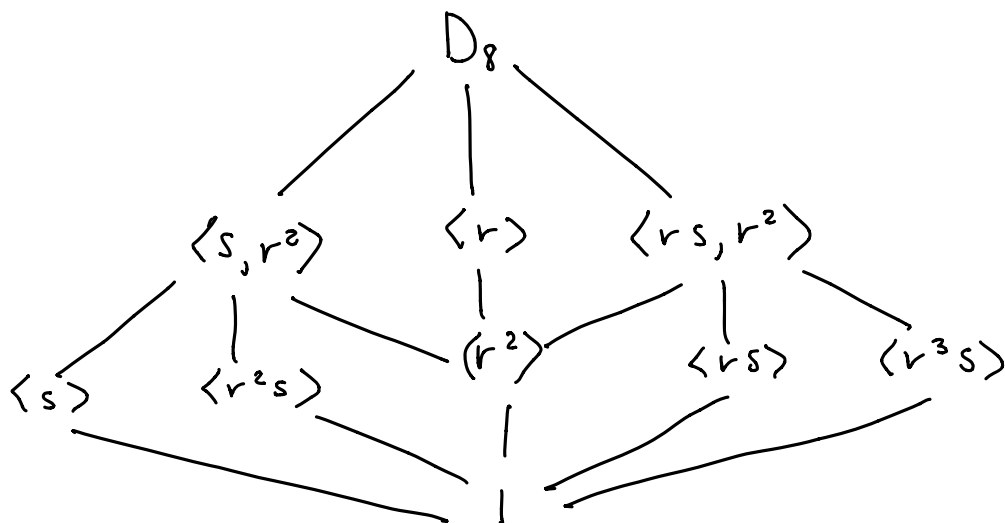


2.) What are the subgroups of  $D_8$ ? They can only have order 1, 2, 4, or 8.

The elts of order 2 are  $r^2, s, sr, sr^2, sr^3$ .

The elts of order 4 are  $r, r^3$ .

The subgroup lattice looks like:



3.)  $Q_8$  is more straightforward, since  $i, j, k$  all have order 4

